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Mean-field theory of sandpiles

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We propose a mean-field theory of sandpiles with dissipation introduced in a clear and physical way. We obtain all exponents for our model by constructing a master equation and mapping the model into a branching process. Two of the exponents are found to depend on a parameter relating the rate of dissipation to that of the addition of sand grains to the system, whereas the others are universal. [S1063-651X(97)07402-3]

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Bak, Tang, and Wiesenfeld [1] introduced the idea of self-organized criticality (SOC) as a paradigm for the ubiquity of spatial and temporal fractals in nature. Since their pioneering work, many models of SOC have been proposed and studied [2–8], analytically, numerically, and experimentally, but the first system shown to have SOC behavior—sandpiles—remains the most important example of SOC. In this paper we construct a mean-field theory of sandpiles. The mean-field theory proposed by us is by no means the first mean-field theory of sandpiles [9–11]. The first mean-field theory of sandpiles was proposed by Tang and Bak [9], who related SOC to conventional critical phenomena. They obtained the master equation for their model and deduced several exponents from it, and then used scaling laws [12] to get the rest of the exponents. The latest development in the mean-field theory of sandpiles is due to Zapperi, Lauritsen, and Stanley [10], who introduced the notion of a self-organized branching process, which allowed them to explicitly include dissipation that was absent in the model of Tang and Bak and, as we show later, is crucial for reaching the SOC state. In our approach, we attempt to combine the best from the previous approaches: we are able to obtain a master equation for our model and map it to a branching process. As a result, (i) we introduce dissipation in our model in a clear and physical way, (ii) we are able to obtain all exponents for our model analytically, (iii) strikingly, some of the exponents are non-universal and depend on a parameter that characterizes the relationship between the rates of dissipation and the external addition of sand grains, and (iv) the universal exponents obtained by us coincide with those obtained previously [9–11].

We consider the following variant of the usual sandpile model: on each of N sites we define an integer number z_i, which represents the number of sand grains at this site. Every time step each z_i increases by 1 with probability $h \ll 1$: $z_i \to z_i+1$. If z_i exceeds $z_c=1$, then at the next time step $z_i \to z_i-2$ and we increase with probability ϵ the height of one randomly chosen site by 1 and with probability $1-\epsilon$ the heights of two randomly chosen sites increased by 1. $\epsilon \ll 1$ is a measure of the dissipation and represents the probability for a site to be on the boundary of the system. The SOC state is reached on first letting $h \to 0$ and then $\epsilon \to 0$.

The dynamics of avalanches is controlled by the average number of sand grains $\Theta = \langle z_i \rangle$. If Θ is greater than the critical value Θ_c, there will be a spontaneous flow (avalanches) even in the absence of the external field h.

One can introduce a number of exponents for this model [9,12]: the distributions of avalanche sizes $D_s(s)$ and durations $D_t(t)$ are both described by power laws $D_s(s) \sim s^{-\tau+1}$ and $D_t(t) \sim t^{-b}$, the cutoff in the avalanche size $s_{\mathrm{co}} \sim |\Theta_c - \Theta|^{-1/\sigma}$, and the cutoff in the avalanche duration $t_{\mathrm{co}} \sim |\Theta_c - \Theta|^{-\nu}$. The duration of an avalanche t is related to its size s through the dynamical exponent [13] z: $t \sim s^z$. To give precise meaning to this statement, one can consider the conditional probability for an avalanche to have duration t, if it has size s: $P(t|s) = f(t/s^z)/t$. [Note that the exact correspondence $t = s^z$ has the scaling form $P(t|s) = \delta(t - s^z) = \delta(1 - s^z/t)/t$.] The average duration of avalanches $\langle t \rangle$ that have size s is $\langle t \rangle = \int_0^\infty t\, P(t|s) dt \sim s^z$. One can relate z to other components: $D_t(t) = \int_0^\infty D_s(s) P(|s) ds \sim t^{(2-\tau-z)/z}$ and thus $b = 1 + (\tau-2)/z$ or $z = (\tau-2)/(b-1)$. From the definitions of σ and ν one gets $z = \nu\sigma$.

The master equation for this model is, as usual, a balance equation, which states that the change in the number of sites of a given height i equals the number of sites that change their height to i minus the number of sites of height i, which change their height from i to some other value:

$$P_i(t+1) - P_i(t) = \sum_{j=0}^{\infty} [P_j(t) T_{ji}(t) - P_i(t) T_{ij}(t)], \quad (1)$$

where $P_i(t)$ is the fraction of sites of height i at time t and T_{ij} is the transition probability for any given site of height i to change its height to j. If one makes an approximation that there are only sites with heights of 0, 1, 2, and 3 and that $P_2 \ll 1$ and $P_3 \ll 1$, then the relevant T_{ij} are

$$T_{01}=T_{12}=T_{21}=T_{32}=hv_0+(1-h)v_1,$$

$$T_{02}=T_{13}=hv_1+(1-h)v_2,$$

$$T_{03}=T_{23}=hv_2,$$

$$T_{20}=T_{31}=(1-h)v_0,$$

$$T_{10}=T_{30}=0,$$

where $v_0=1-A+A^2/2$, $v_1=A-A^2$, $v_2=A^2/2$, and $A=(2-\epsilon)(P_2+P_3)$.

To calculate the transition probabilities T_{ij} one starts by noting that at every time step there are $2(P_2+P_3)N$ grains released as the result of toppling of the active sites; $\epsilon(P_2+P_3)N$ of these grains are carried out of the system and $(2-\epsilon)(P_2+P_3)N=AN$ grains are left. Thus we have $AN \ll N$ grains to be distributed among N sites. The probability v_i for a given site to receive exactly i out of the total AN grains is given by a binomial distribution $v_i=\binom{AN}{i}N^{-i}(1-1/N)^{AN-i}$. v_i rapidly decreases with i and we neglect all v_i with $i>2$. The remaining v_i can be approximated by a Poisson distribution (on assuming that $AN \gg 1$) $v_i=\exp(-A)A^i/i!$. Because $A \ll 1$ we can expand $\exp(-A)$ and obtain the above results for v_i. There are two possibilities for a site with no grains at a given time step to become a site with one grain at the next time step: (i) it can receive one grain from the external field (h) and receive nothing from other sites (v_0) or (ii) it can receive nothing from the external field $(1-h)$ and receive one grain from one of the other sites (v_1). Thus $T_{01}=hv_0+(1-h)v_1$. Similar considerations allow one to obtain all other T_{ij}.

Note also that $\Theta=\langle z_i\rangle=P_1+2P_2+3P_3$. In this approximation one finds that in the steady state $P_1=\frac{1}{2}+O(h^2/\epsilon^2)$, $P_2=h/\epsilon+O(h^2/\epsilon^2)\ll 1$, $P_3=O(h^2/\epsilon^2)\ll P_2$, and $\Theta=\frac{1}{2}+2h/\epsilon+O(h^2/\epsilon^2)$, provided that $h/\epsilon \to 0$. The introduction of higher-order terms P_4, etc., brings in corrections of order $(h/\epsilon)^3$. The result for P_2 can be easily understood: the number of sand grains entering the system at every time step is h and the number of sand grains leaving the system at every time step is ϵP_2; in the steady state these two numbers should be equal. It is important to note that the presence of an $\epsilon>h$ is crucial for reaching the steady state. If $\epsilon<h$, then the dissipation cannot balance the incoming flux and the system will accumulate sand without reaching the steady state. These considerations show that the $h\to 0$ limit must be taken before the $\epsilon \to 0$ limit.

The state of the system is determined by specifying two variables P_1 and P_2 ($P_0=1-P_1-P_2$ and $P_3 \ll P_2$). One can consider relaxation of those variables towards their steady-state values. From the master equation one finds that P_1 and P_2 relax independently, with characteristic times $t_1=\epsilon/4h$ and $t_2=2/\epsilon$. The cutoff in the avalanche durations t_{co} is the smaller of t_1 and t_2. If we assume that $h\sim \epsilon^{1+\mu}$, with $\mu>0$, then $\Theta_c=\frac{1}{2}$, if $\mu \le 1$, then $t_{co}=t_1$ and $\nu=1$, whereas if $\mu>1$, then $t_{co}=t_2$ and $\nu=1/\mu$. Thus $\nu=\min(1,1/\mu)$. The $\mu \le 1$ case leads to exponents that are in agreement with previous results [9,10].

To obtain the values of other exponents we map this model to a branching process: we assume that at any time all P_i take on their steady-state values; thus the number of active sites (sites with two and more grains) is much smaller than the total number of sites N. Under this assumption the probability q_l for a given active site to create l active sites ($0 \le l \le 2$) at the next time step does not depend either on the number of active sites present in the system or on time:

$$q_0=\epsilon(P_0+P_2)(1-h)v_0+(1-\epsilon)(P_0+P_2)^2(1-h)^2v_0^2,$$

$$q_1=\epsilon[P_1+(P_0+P_2)(h+1-v_0)]+2(1-\epsilon)(P_0+P_2)\times(1-h)v_0[P_1+(P_0+P_2)(h+1-v_0)],$$

$$q_2=(1-\epsilon)[P_1+(P_0+P_2)(h+1-v_0)]^2.$$

In this approximation $\Theta=\frac{1}{2}+2h/\epsilon$.

In order to calculate the distribution of avalanche sizes, we used the method of generating functions. If ρ_s is the probability to have an avalanche of size s, then the corresponding generating function is defined as $\rho(x)=\Sigma_{s=0}^{\infty}\rho_s x^s$. It satisfies the equation [14] $\rho(x)=x[q_0+q_1\rho(x)+q_2\rho(x)^2]$, which gives us

$$D_s(s)\equiv\rho_s\sim s^{-3/2}\exp\{-[\epsilon^2/4+4(h/\epsilon)^2]s\}. \tag{2}$$

Thus $\tau=\frac{5}{2}$ and $\sigma=\max(1/2,\mu/2)$.

In order to obtain the distribution of avalanche durations, let a_t be the probability that the avalanche will be over in t or fewer times steps. One can write

$$a_{t+1}=q_0+q_1a_t+q_2a_t^2. \tag{3}$$

Equation (3) can be easily understood: if the initial active site does not create any active sites at the next time step, then the avalanche is over; the probability of this event is q_0. With probability q_1 the initial active sites creates just one active site; then we can consider that created site an initial site for another avalanche, which should die in t or fewer time steps in order for the original avalanche to die in $t+1$ or fewer time steps. Finally, with probability q_2, the initial active site creates two active sites leading to two avalanches that should both die in t or fewer time steps.

On introducing $r_t=1-a_t$ and the average number of active sites created by an active site during one time step $m=q_1+2q_2$, one can rewrite Eq. (3) as

$$r_{t+1}=mr_t-q_2r_t^2, \tag{4}$$

which in the critical state $(m=1)$ leads to

$$D_t(t)\equiv a_t-a_{t-1}\equiv r_{t-1}-r_t\sim t^{-2}, \tag{5}$$

otherwise

$$D_t(t)\equiv a_t-a_{t-1}\equiv r_{t-1}-r_t\sim\exp[-(1-m)t] =\exp[-(\epsilon/2-2h/\epsilon)t], \tag{6}$$

giving $\nu = \min(1, 1/\mu)$, $b = 2$, and $z = (\tau - 2)/(b - 1) = \nu\sigma = \frac{1}{2}$.

All the exponents agree with previous estimates [9–11] on setting $\mu \leqslant 1$. To our knowledge, the nonuniversal values of ν and σ are a new feature of our calculation.

To summarize, we have presented a mean-field theory of a sandpile model with dissipation introduced explicitly and in a physical way. We obtained all exponents for this model by constructing a master equation for it and by mapping the model to a branching process. The exponents, obtained by us for a range of parameter values, agree with mean-field exponents obtained by different means previously [9–11]: $\tau = \frac{5}{2}$, $b = 2$, $\sigma = \frac{1}{2}$, $\nu = 1$, and $z = \frac{1}{2}$. The values of τ, b, and z are found to be universal, while σ and ν depend on a parameter that relates the dissipation rate to the rate characterizing the addition of sand grains.

Note added. After this work was completed, we learned of similar calculations by Lauritsen, Zapperi, and Stanley [15] who used somewhat different techniques. The key difference between their work and ours is that they do not introduce the parameter h. Our results are in accord with theirs where there is overlap.

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[13] Note the nonorthodox definition of z. Typically, the time t and length scale L are related by $t \sim L^z$. Our definition of z is different because the concept of length is not well defined in our model.

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